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1994 J. Phys. A: Math. Gen. 27 3937

(http://iopscience.iop.org/0305-4470/27/11/040)

J. Phys. A: Math. Gen. 27 (1994) 3937–3954. Printed in the UK

Electroviscoelastic Rayleigh–Taylor instability of Maxwell fluids: I. Effect of a constant tangential electric field

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Received 8 November 1993, in final form 1 February 1994

Abstract. The stability of the Rayleigh–Taylor model for an electroviscoelastic Maxwell fluid are investigated. The method of multiple scales is used in order to obtain the stability conditions. A transcendental dispersion relation is obtained at zero-order. The special case, when the two fluids have the same kinematic viscosity, is considered to relax the complexity of the transcendental dispersion relation. The solvability conditions introduce a first-order differential equation. It is found that the increase in the relaxation time λ has a destabilizing influence. Also the increase in the kinematic viscosity in the presence of the parameter λ yields a destabilizing effect. The increase in the kinematic viscosity in the absence of elasticity (pure viscous fluids) has a stabilizing effect.

1. Introduction

In the classical linearized theory of elasticity, the stress in a sheared body is proportional to the amount of shear. In the Navier–Stokes theory of viscosity the shearing stress is proportional to the rate of the shear. In most materials under appropriate circumstances, effects of both elasticity and viscosity are noticeable. If these effects are not further complicated by behaviour that is unlike either elasticity or viscosity, we call the material viscoelastic. The term viscoelastic will be used to describe the properties of any material which, under the appropriate conditions, is able both to store energy in elastic deformation and to dissipate energy as heat. If the strain and the rate of strain are kept sufficiently small so that the ratio of stress to strain is a function of time (or frequency) only and independent of the stress level, the material is said to show linear viscoelastic behaviour. Linear behaviour is easily obtained in dynamic (oscillatory) experiments where the amplitude of deformation is usually extremely small.

The dynamic behaviour of fluids, when the existing molecules are in equilibrium, is disturbed by an applied stress. The stress may be mechanical, shear or compression, or, in the case of polar molecules, it may be applied electrically. The attainment of a new equilibrium state following the stress application is not instantaneous, but takes a finite time which depends on the ability of molecules to move relative to their neighbours. The observed response of the fluid depends on the relative duration of the applied stress compared with the time constant, or relaxation time, which is associated with the change in equilibrium.

The phenomenon of interfacial stability in multilayer flow of viscoelastic fluids is of interest in many polymer processing applications, such as coextrusion of films and fibres. However, the manufacture of these multilayer plastic structures is not without problems. One major problem is the formation of interfacial waves which can result in a significant

0305-4470/94/113937+18$19.50

deterioration of product properties (i.e. mechanical, optical, vapour barrier, etc). Hence, to establish processing windows for stable operation of coextrusion processes, a better understanding of interfacial instabilities is required.

Very few papers have been written on viscoelastic surface waves (Tejero *et al* [1], Borcherdt [2], Currie *et al* [3,4] and Zahorski [5]). Recently, in 1990, Saasen [6,7] investigated the surface gravity waves on a semi-infinite incompressible Maxwell fluid. He found that damped Rayleigh-type gravity waves may exist on a surface of a fluid. He found also that for sufficiently large wavenumber, elasticity promotes travelling waves. His discussion did not include the effect of electrical stress on the stability and his discussion of the dispersion relation is not complete. He plotted the dispersion relation against the wavenumber.

In recent years some experimental studies have addressed the problem of interfacial stability. Yu and Sparrow [8] studied the superposed flow of mineral oil and water in a transparent rectangular duct and showed that viscosity stratification is sufficient to cause interfacial instability even at low Reynolds numbers. However, with their experimental device they were unable to investigate the effect on disturbance wavelength of the stability of the interface. Lee and White [9] studied the interfacial deformation of two superposed viscoelastic fluids in a capillary die and showed the existence of interfacial instabilities in these flows. Han *et al* [10] examined superposed flow of two polymer melts in a slit die. In this investigation the region of interfacial stability was delineated as a function of the interfacial viscosity ratio and layer-depth ratio. Wilson and Khomami [11–13] examined the nature of interfacial deformation and instability in multilayer viscoelastic fluids under various flow kinematics (i.e. shearing, extensional and mixed) in superposed plan Poiseuille flows. They investigated the role of elasticity in interfacial stability, the existence of subcritical and supercritical bifurcations and the combined effects of interfacial instability and layer encapsulation. They found experimentally that elasticity plays a key role in the interfacial instability problem.

The purpose of this work is to examine theoretically the effect of the electric force on the interface separating two semi-infinite Maxwell fluids. The electric field will be treated as a constant electric field. The linear stability theory will be considered here.

2. Formulation of the problem

Consider an interface at the plane $y = 0$ between two semi-infinite fluids. The system is assumed to have a viscoelastic nature described by the Maxwell constitutive relation. Both the fluids are incompressible, isotropic and dielectric. The system is assumed to be stressed by a tangential electric field given by

$$E = E_0 e_x. \tag{1}$$

The system is initially motionless. The motion results from interfacial perturbations. The surface deflection is expressed as

$$y = \xi(x, t). \tag{2}$$

The unit normal vector to the interface is:

$$n = -\frac{\partial \xi}{\partial x} e_x + e_y. \tag{3}$$

The equations which govern the behaviour of a Maxwell fluid are:

$$\frac{\partial}{\partial t}V_j + V_i\frac{\partial}{\partial x_i}V_j = \frac{\partial}{\partial x_i}\sigma_{ij} + \rho F_j \tag{4}$$

$$\frac{\partial}{\partial x_j}V_j = 0 \tag{5}$$

$$\tau_{ij} + \lambda\left(\frac{\partial}{\partial t} + V_k\frac{\partial}{\partial x_k}\right)\tau_{ij} = \mu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \tag{6}$$

where σ_{ij} is the stress tensor, τ_{ij} is the stress deviator $\sigma_{ij} - \pi_{ij}$ and λ is the Maxwell relaxation time. The fluid density and viscosity are denoted by ρ and μ, respectively, $\boldsymbol{V}$ is the velocity vector and $\boldsymbol{F}$ is the body force per unit mass. The electric stress π_{ij} [14] is given by:

$$\pi_{ij} = -\pi\delta_{ij} + \varepsilon E_i E_j - \tfrac{1}{2}\varepsilon E^2\delta_{ij} \tag{7}$$

where π is the modified pressure defined by:

$$\pi = P - \tfrac{1}{2}\varepsilon E_0^2 \tag{8}$$

and where P is the hydrostatic pressure, and ε is the dielectric constant. Equations (4) and (6), in vector form are:

$$\left[1+\lambda\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla\right)\right]\left[\frac{\partial}{\partial t}\boldsymbol{V} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\right] = -\frac{1}{\rho}\left[1+\lambda\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla\right)\right]\nabla\pi + \nu\nabla^2\boldsymbol{V} - g\boldsymbol{e}_y \tag{9}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. The equilibrium state for equation (9) is:

$$\left(1+\lambda^{(r)}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial y}\pi_0^{(r)}(y,t) = \rho^{(r)}g. \tag{10}$$

The superscripts $r = (1)$ and (2) refer to upper and lower fluids, respectively. Integrating equation (10) with respect to y we obtain

$$\pi_0^{(r)} = -\rho^{(r)}gy + \left(1+\lambda^{(r)}\frac{\partial}{\partial t}\right)^{-1}C^{(r)}(t) \tag{11}$$

where $C^{(1)}$ and $C^{(2)}$ are the time-dependent constants of integration.

We assume that the quasi-electrostatic approximation is valid [15]; then Maxwell's equations reduce to:

$$\nabla\cdot(\varepsilon\boldsymbol{E}) = 0 \tag{12}$$

and

$$\nabla\times\boldsymbol{E} = \boldsymbol{0} \qquad \text{or} \qquad \boldsymbol{E} = -\nabla\phi \tag{13}$$

where ϕ is the electrostatic potential.

Two types of boundary conditions suffice to properly constrain the field equations: conditions at an infinite (perpendicular) distance from the system and conditions at the dividing surface. The former express the requirements that the electric field and the velocity vector tend to zero at infinity. Interfacial boundary conditions can be divided into Maxwell's electric conditions, kinematic relations and stress balances.

(a) Maxwell's electric conditions.

(i) The tangential component of the electric field should be continuous at the interface $y = 0$. This leads to

$$\boldsymbol{n} \times (\boldsymbol{E}^{(1)} - \boldsymbol{E}^{(2)}) = 0 \qquad (y = 0). \tag{14}$$

(ii) The normal component of the electric field is continuous across the interface $y = 0$. That is

$$\boldsymbol{n} \cdot (\varepsilon^{(1)}\boldsymbol{E}^{(1)} - \varepsilon^{(2)}\boldsymbol{E}^{(2)}) = 0 \qquad (y = 0). \tag{15}$$

(b) Kinematic relations. The kinematic boundary conditions at the interface of the system are as follows:

(i) The first kinematic relation follows from the assumption that the velocity vector in each of the phases of the system is continuous at the dividing surface. This implies that

$$\boldsymbol{n} \cdot (\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}) = 0 \qquad (y = 0) \tag{16}$$

$$\boldsymbol{n} \times (\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}) = 0 \qquad (y = 0). \tag{17}$$

(ii) An equation expressing the assumed material character of the dividing surface is required. Such an equation is

$$V_y = \frac{\partial \xi}{\partial t} + V_x \frac{\partial \xi}{\partial x} \qquad (y = 0) \tag{18}$$

where V_x and V_y are the tangential and normal velocities, respectively.

(iii) The conditions satisfied by the stress at the interface is

$$(\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)})n_j = -T(\boldsymbol{\nabla} \cdot \boldsymbol{n})n_i \qquad (y = 0). \tag{19}$$

3. Perturbation equations

The linearized equations governing the perturbation quantities are readily found to be

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial \boldsymbol{V}_1}{\partial t} = -\frac{1}{\rho}\left(1 + \lambda \frac{\partial}{\partial t}\right) \boldsymbol{\nabla}\pi_1 + \nu \boldsymbol{\nabla}^2 \boldsymbol{V}_1 \tag{20}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{V}_1 = 0 \tag{21}$$

$$\boldsymbol{\nabla} \cdot (\varepsilon \boldsymbol{E}_1) = 0 \tag{22}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E}_1 = \boldsymbol{0}. \tag{23}$$

Since $\boldsymbol{E}_1$ is a conservative field, then there exists a potential function ϕ_1 such that:

$$\boldsymbol{E}_1 = -\boldsymbol{\nabla}\phi_1 \tag{24}$$

$$\boldsymbol{E} = E_0 \boldsymbol{e}_x - \boldsymbol{\nabla}\phi_1 \tag{25}$$

and hence

$$\nabla^2\phi_1 = 0. \tag{26}$$

If we take the divergence of equation (20) and use the continuity equation (21) we obtain:

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\nabla^2\pi_1(x,y,t) = 0. \tag{27}$$

Operating on both sides of equation (20) by ∇^2, using equation (27) we obtain

$$\nabla^2\left[\nabla^2 - \frac{1}{\nu}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right]V_1 = 0. \tag{28}$$

The linearized boundary conditions are

$$\frac{\partial}{\partial x}[\phi_1^{(1)}(x,y,t) - \phi_1^{(2)}(x,y,t)] = 0 \qquad (y=0) \tag{29}$$

$$ik(\varepsilon^{(1)} - \varepsilon^{(2)})\xi E_0 + \frac{\partial}{\partial y}(\varepsilon^{(1)}\phi_1^{(1)} - \varepsilon^{(2)}\phi_1^{(2)}) = 0 \qquad (y=0) \tag{30}$$

$$V_{1y}^{(1)}(x,y,t) = V_{1y}^{(2)}(x,y,t) = \frac{\partial}{\partial t}\xi(x,t) \qquad (y=0) \tag{31}$$

$$\frac{\partial}{\partial y}[V_{1y}^{(1)}(x,y,t) - V_{1y}^{(2)}(x,y,t)] = 0 \qquad (y=0). \tag{32}$$

The continuity of the normal stress σ_{ij} at the interface $y=0$ requires that:

$$n_y(\sigma_{iy}^{(1)} - \sigma_{iy}^{(2)}) = -n_i T\nabla^2\xi \qquad (y=0) \tag{33}$$

where T is the surface tension through the surface separating fluids. The stress tensor σ_{ij} is given by:

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\sigma_{ij} = \left(1+\lambda\frac{\partial}{\partial t}\right)(-\pi\delta_{ij} + \varepsilon E_i E_j - \tfrac{1}{2}\varepsilon E^2\delta_{ij}) + \mu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right). \tag{34}$$

Substituting from (11) into (33), using (34), the result is

$$\pi_1^{(1)} - \pi_1^{(2)} + \tilde{\sigma}\xi - E_0\left[\varepsilon^{(1)}\frac{\partial\phi_1^{(1)}}{\partial x} - \varepsilon^{(2)}\frac{\partial\phi_1^{(2)}}{\partial x}\right]$$
$$= 2\nu^{(1)}\rho^{(1)}\left(1+\lambda^{(1)}\frac{\partial}{\partial t}\right)^{-1}\frac{\partial V_{1y}^{(1)}}{\partial y} - 2\nu^{(2)}\rho^{(2)}\left(1+\lambda^{(2)}\frac{\partial}{\partial t}\right)^{-1}\frac{\partial V_{1y}^{(2)}}{\partial y} \qquad (y=0) \tag{35}$$

where

$$\tilde{\sigma} = k^2T - (\rho^{(1)} - \rho^{(2)})g.$$

The continuity of the tangential stress across the interface $y = 0$ requires that:

$$\nu^{(1)}\rho^{(1)}\left(1+\lambda^{(1)}\frac{\partial}{\partial t}\right)^{-1}\left(\frac{\partial V_{1x}^{(1)}}{\partial y}+\frac{\partial V_{1y}^{(1)}}{\partial x}\right) - \nu^{(2)}\rho^{(2)}\left(1+\lambda^{(2)}\frac{\partial}{\partial t}\right)^{-1}\left(\frac{\partial V_{1x}^{(2)}}{\partial y}+\frac{\partial V_{1y}^{(2)}}{\partial x}\right)=0 \qquad (y=0).$$

Using the continuity equation (21) we obtain:

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\left[\nu^{(1)}\rho^{(1)}\left(1+\lambda^{(1)}\frac{\partial}{\partial t}\right)^{-1}V_{1y}^{(1)}-\nu^{(2)}\rho^{(2)}\left(1+\lambda^{(2)}\frac{\partial}{\partial t}\right)^{-1}V_{1y}^{(2)}\right]=0 \quad (y=0). \tag{36}$$

In order to obtain a travelling-wave solution we may assume the following dependence:

$$\xi(x,t)=\gamma(t)\mathrm{e}^{ikx} \tag{37}$$

$$\phi_1(x,y,t)=\hat{\phi}(y,t)\mathrm{e}^{ikx} \tag{38}$$

$$V_1(x,y,t)=\hat{V}(y,t)\mathrm{e}^{ikx} \tag{39}$$

$$\pi_1(x,y,t)=\hat{\pi}(y,t)\mathrm{e}^{ikx} \tag{40}$$

where k is the wavenumber, which is assumed to be positive. Substituting expansion (38) into equation (26), the solution of the resulting differential equation leads to:

$$\phi_1(x,y,t)=[C_1(t)\mathrm{e}^{-ky}+C_2(t)\mathrm{e}^{ky}]\mathrm{e}^{ikx}. \tag{41}$$

Since $\phi_1(x,y,t)$ must be finite as $y\to\pm\infty$, equation (41) reduces to:

$$\phi_1^{(1)}(x,y,t)=C_1(t)\mathrm{e}^{ikx-ky} \tag{42}$$

$$\phi_1^{(2)}(x,y,t)=C_2(t)\mathrm{e}^{ikx+ky}. \tag{43}$$

Substituting from expansions (42) and (43) into equations (29) and (30), we obtain:

$$\phi_1^{(1)}(x,y,t)=\mathrm{i}E_0\gamma(t)\frac{(\varepsilon^{(1)}-\varepsilon^{(2)})}{(\varepsilon^{(1)}+\varepsilon^{(2)})}\mathrm{e}^{ikx-ky} \tag{44}$$

$$\phi_1^{(2)}(x,y,t)=\mathrm{i}E_0\gamma(t)\frac{(\varepsilon^{(1)}-\varepsilon^{(2)})}{(\varepsilon^{(1)}+\varepsilon^{(2)})}\mathrm{e}^{ikx+ky}. \tag{45}$$

Substituting from (39) into equation (20), the resulting y-component is:

$$\left(1+\lambda^{(r)}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial y}\hat{\pi}^{(r)}(y,t)=\rho^{(r)}\left[\nu^{(r)}\left(\frac{\partial^2}{\partial y^2}-k^2\right)-\left(1+\lambda^{(r)}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right]\hat{V}_y^{(r)}(y,t). \tag{46}$$

If we substitute from (39) into equation (28), the resulting y-component is:

$$\left(\frac{\partial^2}{\partial y^2}-k^2\right)\left[\nu^{(r)}\left(\frac{\partial^2}{\partial y^2}-k^2\right)-\left(1+\lambda^{(r)}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right]\hat{V}_y^{(r)}(y,t)=0 \tag{47}$$

where $V_{1y}^{(r)}(x,y,t)=\hat{V}_y^{(r)}(y,t)\mathrm{e}^{ikx}$ is the normal velocity of (39).

This problem has been treated particularly for viscous fluids in absence of elasticity and electric field by Chandrasekhar [18]. The additional problem of viscous fluid influenced by constant electric field has been treated by Melcher and Scharz [20].

4. The solution using the method of multiple time-scales

The complexity of the mathematical problem forces us to use a perturbation technique. We use the method of multiple time-scales in order to carry out the stability analysis of the problem [16]. In applying the method of multiple time-scales, we may use the time-scales T_0 and T_1 so that:

$$T_0 = t \qquad \text{and} \qquad T_1 = \tilde{\varepsilon} t$$

where $\tilde{\varepsilon}$ is a small dimensionless parameter defined as:

$$\lambda^{(1)} = \tilde{\varepsilon}\tilde{\lambda}^{(1)} \qquad \text{and} \qquad \lambda^{(2)} = \tilde{\varepsilon}\tilde{\lambda}^{(2)}$$

with finite $\tilde{\lambda}^{(1),(2)}$.

Because the vanishing of the Maxwell relaxation time λ from the equation of motion (9) leads to the Navier–Stokes equation, the small dimensionless parameter $\tilde{\varepsilon}$ is introduced such that when $\tilde{\varepsilon} = 0$ the fluid behaves as a purely viscous fluid. This is the reason to choose $\tilde{\varepsilon}$ as a small magnitude of the relaxation time.

One assumes that the solution of equation (47) can be represented by an expansion having the form:

$$\hat{V}_y(y, t, \tilde{\varepsilon}) = \hat{V}_0(y, T_0, T_1) + \tilde{\varepsilon}\hat{V}_1(y, T_0, T_1) + \tilde{\varepsilon}^2\hat{V}_2(y, T_0, T_1) + \cdots. \tag{48}$$

Also, we may consider the following expansions:

$$\gamma(t, \tilde{\varepsilon}) = \gamma_0(T_0, T_1) + \tilde{\varepsilon}\gamma_1(T_0, T_1) + \tilde{\varepsilon}^2\gamma_2(T_0, T_1) + \cdots \tag{49}$$

$$\hat{\pi}(y, t, \tilde{\varepsilon}) = \hat{\pi}_0(y, T_0, T_1) + \tilde{\varepsilon}\hat{\pi}_1(y, T_0, T_1) + \tilde{\varepsilon}^2\hat{\pi}_2(y, T_0, T_1) + \cdots \tag{50}$$

$$\hat{\phi}(y, t, \tilde{\varepsilon}) = \hat{\phi}_0(y, T_0, T_1) + \tilde{\varepsilon}\hat{\phi}_1(y, T_0, T_1) + \tilde{\varepsilon}^2\hat{\phi}_2(y, T_0, T_1) + \cdots. \tag{51}$$

The scheme assumes that the fluid response is a perturbation about the Newtonian state (viscous fluid) [17]. The perturbation scheme is applied to analyse the effects of the elasticity when balanced with the viscosity under the effect of an electric field.

Substituting from the expansions (48), (49), (50) and (51) into the equations of motion (46) and (47) and the boundary conditions (31), (32), (35) and (36), and equating the coefficients of equal powers of $\tilde{\varepsilon}$, we obtain equations of zero-order and first-order in $\tilde{\varepsilon}$.

5. The zero-order problem

In deriving the characteristic equation, in the zero-order problem, we may assume the solution of zero-order equations as follows:

$$\gamma_0(T_0, T_1) = \gamma_{00}(T_1)e^{\sigma_0 T_0} \tag{52}$$

$$\hat{V}_0(y, T_0, T_1) = \hat{V}_{00}(y, T_1)e^{\sigma_0 T_0} \tag{53}$$

$$\hat{\pi}_0(y, T_0, T_1) = \hat{\pi}_{00}(y, T_1)e^{\sigma_0 T_0} \tag{54}$$

where σ_0 is the frequency of the disturbance. Thus the distribution of the velocity and pressure in the two phases are obtained to be

$$\hat{V}_0^{(1)}(y, T_0, T_1) = [(\sigma_0 - 2A_2)e^{-ky} + 2A_2 e^{-km_1 y}]\gamma_{00}(T_1)e^{\sigma_0 T_0} \qquad (y > 0) \tag{55}$$

$$\hat{V}_0^{(2)}(y, T_0, T_1) = [(\sigma_0 - 2A_1)e^{ky} + 2A_1 e^{km_2 y}]\gamma_{00}(T_1)e^{\sigma_0 T_0} \qquad (y < 0) \tag{56}$$

$$\hat{\pi}_0^{(1)}(y, T_0, T_1) = \frac{\rho^{(1)}\sigma_0}{k}(\sigma_0 - 2A_2)\gamma_{00}(T_1)e^{\sigma_0 T_0 - ky} \qquad (y > 0) \tag{57}$$

$$\hat{\pi}_0^{(2)}(y, T_0, T_1) = \frac{\rho^{(2)}\sigma_0}{k}(\sigma_0 - 2A_1)\gamma_{00}(T_1)e^{\sigma_0 T_0 + ky} \qquad (y < 0) \tag{58}$$

where

$$m_j^2 = 1 + \frac{\sigma_0}{k^2 \nu^{(j)}} \tag{59}$$

$$A_j = \left[\frac{\rho^{(j)}\sigma_0 + (-1)^{j+1}k^2(1 - m_j)(\nu^{(1)}\rho^{(1)} - \nu^{(2)}\rho^{(2)})}{\rho^{(1)}(1 - m_2) + \rho^{(2)}(1 - m_1)}\right]. \tag{60}$$

With the following dispersion relation:

$$4k^4(m_1-1)(m_2-1)(\nu^{(1)}\rho^{(1)}-\nu^{(2)}\rho^{(2)})^2 - 4k^2\sigma_0(\nu^{(1)}\rho^{(1)}-\nu^{(2)}\rho^{(2)})[\rho^{(1)}(m_2-1)-\rho^{(2)}(m_1-1)]$$
$$-[\rho^{(1)}(m_2-1)+\rho^{(2)}(m_1-1)][k^2E_0^2\varepsilon^* + k\tilde{\sigma} + \sigma_0^2(\rho^{(1)}+\rho^{(2)})] - 4\rho^{(1)}\rho^{(2)}\sigma_0^2 = 0 \tag{61}$$

where $\varepsilon^* = \frac{(\varepsilon^{(1)}-\varepsilon^{(2)})^2}{\varepsilon^{(1)}+\varepsilon^{(2)}}$.

In the limit of no electric field, the dispersion equation (61) is reduced to that discussed by Chandrasekhar [18].

6. The first-order problem

The knowledge of the Newtonian problem obtained in the zero-order problem is sufficient to determine the solution at this order. The solvability condition corresponds to terms containing the factor of $e^{\sigma_0 T_0}$. Hence the solution of the above set of equations in terms of $e^{\sigma_0 T_0}$ is given by:

$$\hat{V}_{11}^{(1)}(y, T_0, T_1) = \left\{[D_1(1 - C_2) - B_2]e^{-ky} + \left[(B_2 + C_2 D_1) - \frac{y}{km_1\nu^{(1)}}(D_1 + \tilde{\lambda}^{(1)}\sigma_0^2)A_2\right]e^{-km_1 y}\right\}\gamma_{00}(T_1)e^{\sigma_0 T_0} \quad (y > 0) \tag{62}$$

$$\hat{V}_{11}^{(2)}(y, T_0, T_1) = \left\{[D_1(1 - C_1) - B_1]e^{ky} + \left[(B_1 + C_1 D_1) - \frac{y}{km_2\nu^{(2)}}(D_1 + \tilde{\lambda}^{(2)}\sigma_0^2)A_1\right]e^{km_2}\right\}\gamma_{00}(T_1)e^{\sigma_0 T_0} \quad (y > 0) \tag{63}$$

$$\hat{\pi}_{11}^{(1)}(y, T_0, T_1) = \frac{\rho^{(1)}}{k}[\sigma_0 D_1(2 - C_2) - 2D_1A_2 - \sigma_0 B_2]\gamma_{00}(T_1)e^{-ky+\sigma_0 T_0} \qquad (y < 0) \tag{64}$$

$$\hat{\pi}_{11}^{(2)}(y, T_0, T_1) = \frac{-\rho^{(2)}}{k}[\sigma_0 D_1(2 - C_1) - 2D_1A_1 - \sigma_0 B_1]\gamma_{00}(T_1)e^{ky+\sigma_0 T_0} \qquad (y < 0) \tag{65}$$

where

$$B_j = \frac{(-1)^j 2k^2\sigma_0(1-m_j)(\rho^{(1)}\nu^{(1)}\tilde{\lambda}^{(1)} - \rho^{(2)}\nu^{(2)}\tilde{\lambda}^{(2)})}{\rho^{(1)}(1-m_2)+\rho^{(2)}(1-m_1)} + \frac{\rho^{(j)}\sigma_0^2}{k^2[\rho^{(1)}(1-m_2)+\rho^{(2)}(1-m_1)]}\left[\frac{\tilde{\lambda}^{(j)}A_{3-j}}{\nu^{(j)}m_j} + \frac{\tilde{\lambda}^{(3-j)}A_j}{\nu^{(3-j)}m_{3-j}}\right] \tag{66}$$

$$C_j = \frac{A_j[\rho^{(j)}\sigma_0 - 2k^2\rho^{(3-j)}\nu^{(3-j)}m_{3-j}(1-m_j)]}{k^2 m_{3-j}\nu^{(3-j)}\sigma_0[\rho^{(1)}(1-m_2)+\rho^{(2)}(1-m_1)]} + \frac{2A_j}{\sigma_0} + \frac{A_{3-j}[\rho^{(j)}\sigma_0 + 2k^2\rho^{(j)}\nu^{(j)}m_j(1-m_j)]}{k^2 m_j\nu^{(j)}\sigma_0[\rho^{(1)}(1-m_2)+\rho^{(2)}(1-m_1)]} \qquad j = 1, 2. \tag{67}$$

Thus, to first-order in $\tilde{\varepsilon}$, the velocity $V_{11}^{(1),(2)}$ and the pressure $\pi_{11}^{(1),(2)}$ complete the set of solutions. Substituting the above solutions into the first order in $\tilde{\varepsilon}$ of the boundary condition (35), the resulting differential equation (i.e the solvability condition) is:

$$[(p_1 + iq_1)D_1 + (p_2 + iq_2)/\tilde{\varepsilon}]\gamma_{00}(T_1) = 0.$$

In terms of the original variable (t), we obtain:

$$\left[(p_1 + iq_1)\frac{d}{dt} + (p_2 + iq_2)\right]\gamma(t) = 0 \tag{68}$$

where the complex coefficients of the above equation depend on the nature of the roots of the dispersion relation (61). These coefficients are given in the following:

$$p_1 + iq_1 = 2k^2\rho^{(1)}\nu^{(1)}[C_2(m_1-1) - B_2 + 1] + 2k^2\rho^{(2)}\nu^{(2)}[C_1(m_2-1) - B_1 + 1] + 2(\rho^{(1)}+\rho^{(2)})\sigma_0 - (\rho^{(1)}C_2 + \rho^{(2)}C_1)\sigma_0 - 2(\rho^{(1)}A_2 + \rho^{(2)}A_1) + \frac{2}{m_1 m_2}(\rho^{(2)}m_1A_1 + \rho^{(1)}m_2A_2) \tag{69}$$

$$p_2 + iq_2 = 2k^2\rho^{(1)}\nu^{(1)}[B_2(m_1-1) - \tilde{\lambda}^{(1)}\sigma_0^2 - \tilde{\lambda}^{(1)}\sigma_0(m_1-1)A_2] + 2k^2\rho^{(2)}\nu^{(2)}[B_1(m_2-1) - \tilde{\lambda}^{(2)}\sigma_0^2 - \tilde{\lambda}^{(2)}\sigma_0(m_2-1)A_1] + \frac{2\sigma_0^2}{m_1m_2}(\rho^{(1)}\tilde{\lambda}^{(1)}m_2A_2 + \rho^{(2)}\tilde{\lambda}^{(2)}m_1A_1) - \sigma_0(\rho^{(1)}B_2 + \rho^{(2)}B_1). \tag{70}$$

Since equation (68) is a first-order differential equation with complex coefficient, then the necessary and sufficient condition for stability is:

$$p_1p_2 + q_1q_2 > 0. \tag{71}$$

7. Special case

Due to the complexity of equation (61) we will restrict our discussion of stability to the case when the kinematic viscosities of the two fluids are the same, i.e when $\nu^{(1)} = \nu^{(2)} = \nu$, consequently $m_1 = m_2 = m$. This assumption simplifies the dispersion equation (61) to the form:

$$(m-1)\{8k^4\nu^2(\rho^{(1)}-\rho^{(2)})^2 + 4k^2\nu\sigma_0(\rho^{(1)}-\rho^{(2)})^2 + k^2E_0^2\varepsilon^* + k\tilde{\sigma})(\rho^{(1)}+\rho^{(2)})\} \\ + m\sigma_0^2(\rho^{(1)}+\rho^{(2)})^2 - \sigma_0(\rho^{(1)}-\rho^{(2)})^2(\sigma_0+4k^2\nu) = 0. \tag{72}$$

A dimensionless length and time will be introduced to present equation (72) in dimensionless form by using the characteristic length $L = (T/\rho^{(2)}g)^{1/2}$ and the characteristic time $t_0 = (L/g)^{1/2}$. Other dimensionless quantities are given by:

$$k = k^*/L \qquad \sigma_0 = \sigma^*/t_0 \qquad \nu = \nu^* L^2/t_0 \qquad E_0^2 = E_0^{*^2}\frac{L}{T}.$$

The characteristic equation is then

$$(m^*-1)[8k^{*^4}\nu^{*^2}\rho^* + 4k^{*^2}\sigma^*\rho^*\nu^* + k^{*^3} + k^*(1-\rho) + k^{*^2}\varepsilon^*E_0^{*^2}] \\ + m^*(1+\rho)\sigma^{*^2} - \rho^*\sigma^*(\sigma^* + 4k^{*^2}\nu^*) = 0 \tag{73}$$

where

$$\rho^* = \frac{(1-\rho)^2}{1+\rho} \qquad \rho = \rho^{(1)}/\rho^{(2)}$$

and

$$m^{*^2} = 1 + \frac{\sigma^*}{k^{*^2}\nu^*}. \tag{74}$$

Due to the transcendental equation (73) the square of it must be calculated in order to take a polynomial form and then for the stability analysis to be available for the problem. In the squaring process unrelated solutions will also appear and a fourth-degree polynomial is obtained:

$$\sigma^{*^4} + a_3\sigma^{*^3} + a_2\sigma^{*^2} + a_1\sigma^* + a_0 = 0 \tag{75}$$

where

$$a_3 = \frac{8k^{*^2}\nu^*}{(1+\rho)^2}[(1-\rho^2)^2 + \rho(1+\rho^2)]$$

$$a_2 = \frac{2k^*}{(1+\rho)}[k^*\varepsilon^*E_0^{*^2} + k^{*^2} + (1-\rho) + 12k^{*^3}\nu^{*^2}\rho^*]$$

$$a_1 = \frac{2k^{*^3}\nu^*}{(1+\rho)^2}\{[3\rho^* + (1+\rho)][k^*\varepsilon^*E_0^{*^2} + k^{*^2} + (1-\rho)] + 8k^{*^3}\nu^{*^2}(1-\rho^2)\}$$

$$a_0 = \frac{k^{*^2}}{(1+\rho)^2}[k^*\varepsilon^*E_0^{*^2} + k^{*^2} + (1-\rho)][k^*\varepsilon^*E_0^{*^2} + k^{*^2} + (1-\rho) + 8k^{*^2}\nu^{*^2}\rho^*].$$

According to the assumption of $\nu^{(1)} = \nu^{(2)} = \nu$, the solvability condition (68) reduces to:

$$\left[(p_1^* + \mathrm{i}q_1^*)\frac{\mathrm{d}}{\mathrm{d}t} + (p_2^* + \mathrm{i}q_2^*)\right]\gamma(t) = 0. \tag{76}$$

The quantities p_1^*, p_2^*, q_1^* and q_2^* are real and are given in the appendix. Here the stability condition (71) reduces to:

$$p_1^*p_2^* + q_1^*q_2^* > 0. \tag{77}$$

8. Numerical results

The characteristic equation (75) controls the stability in the zero-order problem, i.e the case of pure viscosity. The stability conditions in this case will be obtained as follows.

Each negative root (or complex root with a negative real part) of equation (75) corresponds to a stable mode of the interfacial disturbance. Using the Hurwitz criterion for stability, the necessary and sufficient condition for interfacial stability (for all the roots or to have negative real parts) are

$$\text{(i)} \qquad a_1 > 0 \qquad i = 0, 1, 2, 3 \tag{78}$$

$$\text{(ii)} \qquad a_1(a_2a_3 - a_1) - a_0a_3^2 > 0. \tag{79}$$

The above conditions for stability can be summarized as two conditions [19]:

$$a_0 > 0 \qquad \text{and} \qquad a_1(a_2a_3 - a_1) - a_0a_3^2 > 0. \tag{80}$$

The first condition, $a_0 > 0$, is trivially satisfied when $\rho < 1$, i.e when the lower fluid is heavier than the upper one, which characterizes a variety of physical situations. In the case of $\rho > 1$, stability occurs in the presence of the electric field. Condition $a_0 > 0$ is satisfied when

$$E_0^{*^2} > \tilde{E}_1 \qquad \text{or} \qquad E_0^{*^2} < \tilde{E}_2$$

where

$$\tilde{E}_1 = \frac{(\rho - 1) - k^{*^2}}{k^* \varepsilon^*} \tag{81}$$

and

$$\tilde{E}_2 = \frac{(\rho - 1) - k^{*^2} - 8k^{*^3} \nu^{*^2} \rho^*}{k^* \varepsilon^*}. \tag{82}$$

Note that $\tilde{E}_1 > \tilde{E}_2$. The inequality (79) can be rearranged in terms of $E_0^{*^2}$ to become:

$$E_0^{*^4} + b_1 E_0^{*^2} + b_0 < 0 \tag{83}$$

where

$$b_1 = -\frac{512k^{*^2}\nu^{*^2}\rho^*(2\rho^8 + 4\rho^7 - \rho^6 - 2\rho^5 + 4\rho^4 - 2\rho^3 - \rho^2 + 4\rho + 2)}{\varepsilon^*(1+\rho)^4[\rho^{*^2} - (1+\rho)^2]^2} + \frac{2(k^{*^2} - \rho + 1)}{k^*\varepsilon^*} \tag{84}$$

$$b_0 = -\frac{512k^*\nu^{*^2}\rho^*[k^{*^2} + (1-\rho)](2\rho^8 + 4\rho^7 - \rho^6 - 2\rho^5 + 4\rho^4 - 2\rho^3 - \rho^2 + 4\rho + 2)}{\varepsilon^{*^2}(1+\rho)^4[\rho^{*^2} - (1+\rho)^2]^2}$$
$$- \frac{256k^{*^4}\nu^{*^4}\rho^{*^2}}{\varepsilon^{*^2}[\rho^{*^2} - (1+\rho)^2]^2}(11\rho^4 + 8\rho^3 - 30\rho^2 + 8\rho + 11) + \frac{1}{k^{*^2}\varepsilon^{*^2}}[k^{*^2} + (1-\rho)]^2. \tag{85}$$

The equality of the relation (83) is a quadratic equation in $E_0^{*^2}$ which has two roots $\tilde{E}_3$ and $\tilde{E}_4$, given by:

$$\tilde{E}_{3,4} = \tfrac{1}{2}\left[-b_1 \pm \sqrt{b_1^2 - 4b_0}\right]. \tag{86}$$

If the roots $\tilde{E}_3$ and $\tilde{E}_4 (\tilde{E}_3 > \tilde{E}_4)$ are to be real, the discriminant in equation (83) should be positive. Substituting from (84) and (85) into the discriminant, we get:

$$b_1^2 - 4b_0 = \frac{512}{(1+\rho)^2}(2\rho^8 + 4\rho^7 - \rho^6 - 2\rho^5 + 4\rho^4 - 2\rho^3 - \rho^2 + 4\rho + 2)^2$$
$$+ 2[\rho^{*^2} - (1-\rho)^2]^2[11(1-\rho^2)^2 + 8\rho(1-\rho)^2 + 8\rho^2] > 0$$

which is automatically satisfied for all values of ρ. Relation (79) is satisfied if

$$(E_0^{*^2} - \tilde{E}_3)(E_0^{*^2} - \tilde{E}_4) < 0$$

which may be reduced to the following requirement:

$$\tilde{E}_4 < E_0^{*^2} < \tilde{E}_3. \tag{87}$$

From the above discussion we see that the system is stable provided that the electric field $E_0^{*^2}$ satisfies either of the following conditions:

$$E_0^{*^2} > \tilde{E}_1 \qquad \text{and} \qquad \tilde{E}_4 < E_0^{*^2} < \tilde{E}_3$$

or

$$E_0^{*^2} < \tilde{E}_2 \qquad \text{and} \qquad \tilde{E}_4 < E_0^{*^2} < \tilde{E}_3.$$

In figures 1 and 2 we plot $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$ and $\tilde{E}_4$ against k^* for a system having $\rho = 2.5$, i.e the system is statically unstable. Two different cases are considered for which $\nu^* = 0.1$ and 0.5. To show the effect of the kinematic viscosity on the stability, we observe that the increase in stable regions is associated with the increase of the values of ν^*. The presence of viscosity leads to the existence of a stable region. The electric field controls the boundaries of the stable region. The increase of the field decreases the width of the stable ranges of k^*. In the limit as $E_0^* \to 0$, calculations show that stability is possible except for values of $k^* < 0.707107$ for all values of ν^*. For very small values of the wavenumber, very large values of ν^* are needed to express the instability of the system.

In figure 3 the system has $\rho = 0.1$ (i.e. the system is statically stable) and two different ν^* ($\nu^* = 0.1$ and $\nu^* = 0.3$). The area labelled by the symbol S_1 refers to the case of $\nu^* = 0.1$. When ν^* changes to $\nu^* = 0.3$ an additional stable area S_2 is presented. Hence the decrease in the values of ν^* decreases the stable regions. Thus the tangential electric field has a destabilizing effect contrary to the case of inviscid fluids. This result was also obtained by Melcher [20] in dealing with a different model for viscous fluids.

The stability condition (77) is plotted in the $(k^* - \varepsilon^* E_0^{*^2})$ plane for some sample cases. We may note that the RHS of condition (77) depends on σ^*, which is determined by equation (75). Since equation (75) has four roots, it is necessary that every root should satisfy condition (77) independently. The regions in the $(k^* - \varepsilon^* E_0^{*^2})$ plane labelled by (S)

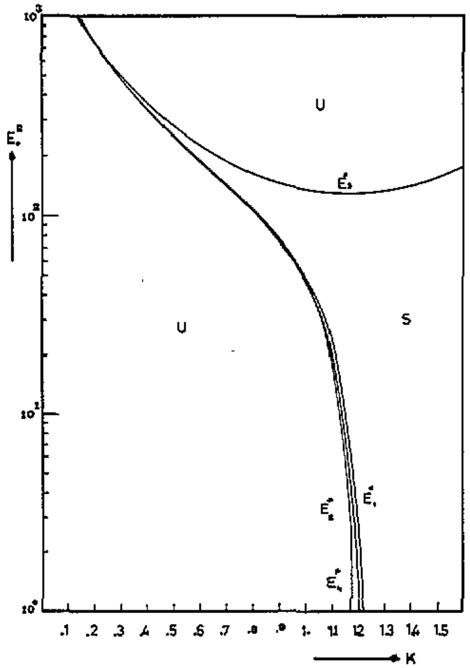

Figure 1. Represents the plane $(k - E_0^2)$ for a system having $\rho = 2.5$, $\varepsilon^* = 0.01$ and $\nu^* = 0.1$. The curves E_i^* $(i = 1, 2, 3, 4)$ are given by the relations (81), (82) and (86). The symbol (S) refers to the stable regions and (U) denotes unstable regions.

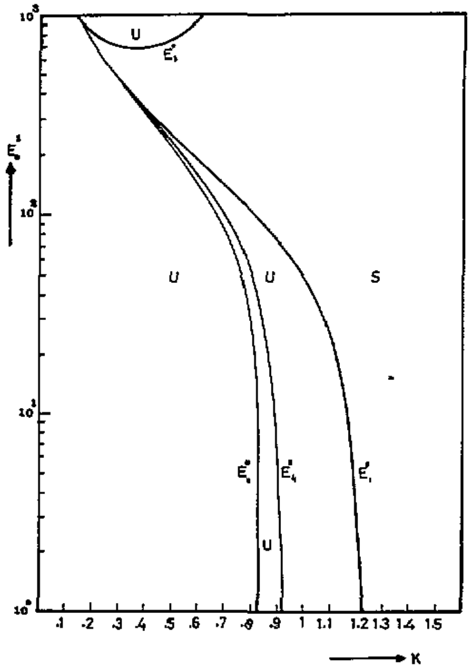

Figure 2. Represents the same system considered in figure 1, but $\nu^* = 0.5$.

represent situations where every point in the region satisfied inequality (77) for the four values of σ^*.

Figure 4 represents a system for $\rho = 0.1$, $\nu^* = 0.5$ and $\lambda_0 (= \lambda^{(1)}/\lambda^{(2)}) = 0.1$. The limit when $\varepsilon^* E_0^{*^2} \to 0$ is calculated numerically. It is found that the two branches bounding (S) intersect the line $\varepsilon^* E_0^{*^2} = 0$. This means that the system is unstable (though $\rho = 0.1 < 1$, i.e statically stable). Figure 4 shows that the presence of the field produces a stable region. Thus the field has a stabilizing effect. The increase in the field changes the area of stability, which continues to increase until there is a critical value in the field ($\varepsilon^* E_0^{*^2} = 260.7$) and then the area decreases with the increase in the field. Figure 5 represents the same system, except that λ_0 is increased to the value $\lambda_0 = 1$. Comparing the two figures we observe that the increase of λ_0 for fixed ν^* has a destabilizing influence. Figure 6 represents the same system considered in figure 4 but ν^* is changed to the value $\nu^* = 1.5$ and λ_0 is changed to the value $\lambda_0 = 1$. It is observed that the increase in the kinematic viscosity in the presence of the elasticity yields a destabilizing effect. Figures 7 and 8 represent a system where $\rho = 1.5 > 1$. This means that the system was statically unstable. Thus larger values of the field are required for stability. In figure 7 the minimum value required for the field is $\varepsilon^* E_0^{*^2} = 23$, while $\varepsilon^* E_0^{*^2} = 21.8$.

9. Conclusion

With the use of the method of multiple time-scales, we perform an investigation of the Rayleigh–Taylor problem of interfacial stability in a two-layer system of electroviscoelastic

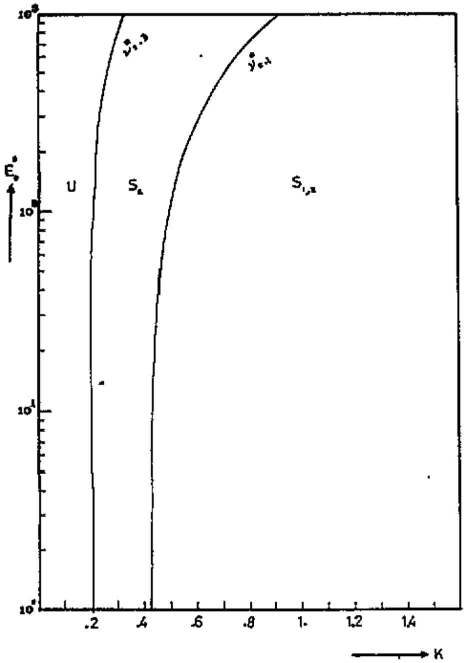

Figure 3. Represents the system having $\rho = 0.1$, $\varepsilon^* = 0.01$ and $\nu^* = 0.1$ and 0.3.

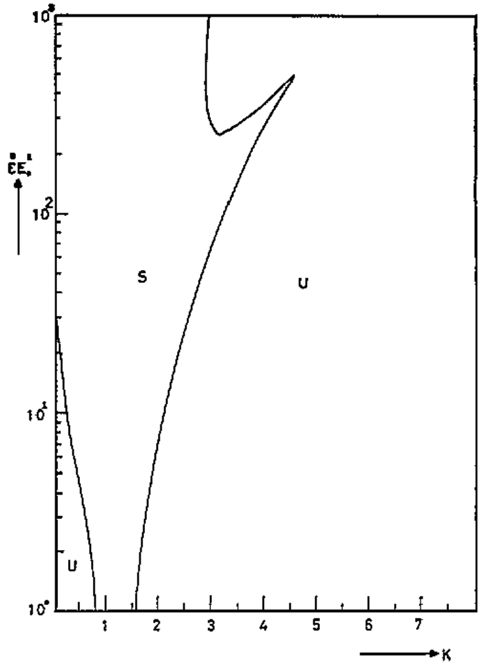

Figure 4. Represents a stable diagram for a system having $\rho = 0.1$, $\nu^* = 0.5$ and $\lambda_0 = 0.1$. The $(k - \varepsilon^* E_0^2)$ plane is plotted from the stability condition (75).

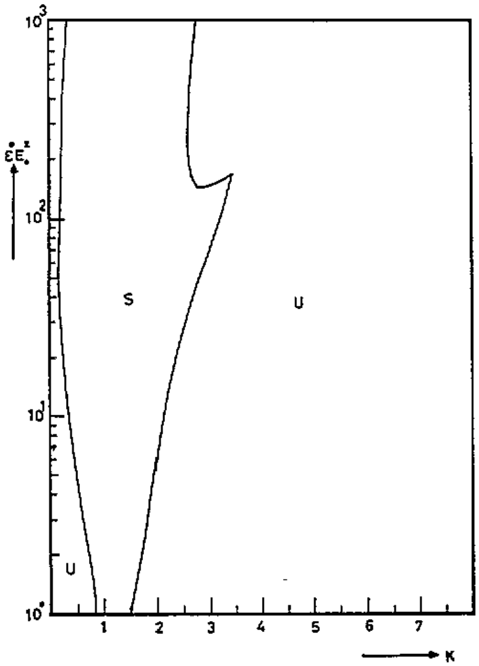

Figure 5. Represents the same system considered in figure 4, but $\lambda_0 = 1$.

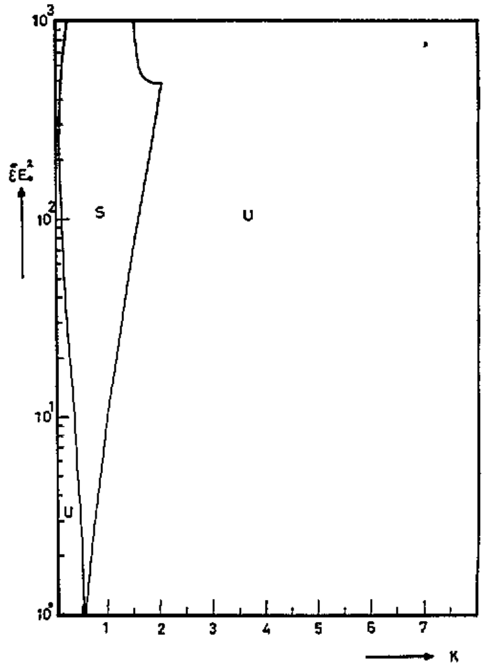

Figure 6. Represents the same system considered in figure 4, but $\nu^* = 1.5$ and $\lambda_0 = 1$.

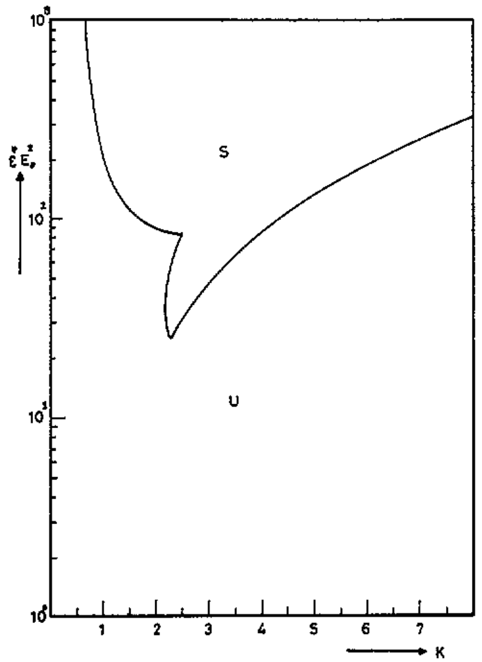

Figure 7. Represents the same system considered in figure 4, but $\rho = 1.5$ and $\lambda_0 = 1$.

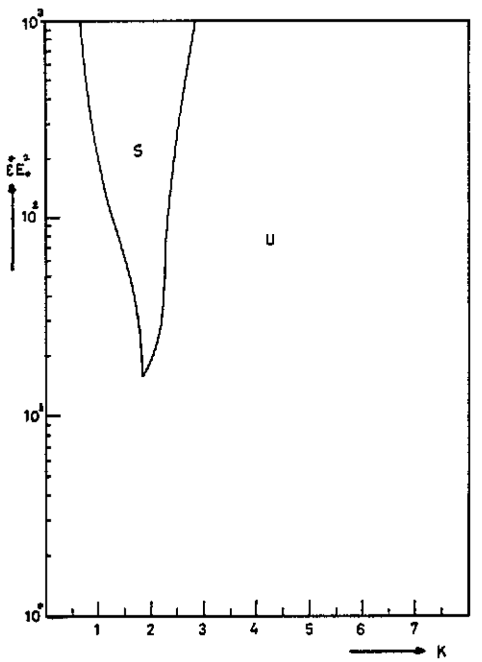

Figure 8. Represents the same system considered in figure 4, but $\rho = 1.5$ and $\lambda_0 = 10$.

Maxwell fluids. We examine the effects on the stability of the interface by applying a constant electric force which is tangential to the interface separating the two fluids.

The stability analysis has been based on the linear perturbation theory. Through the linear perturbation analysis we obtain a fourth-order partial differential equation, which governs the motion of linear viscoelastic fluids. The motivation for using perturbation technique, such as a method of multiple-scales approach, based on the fact of the Maxwell fluid nature, has a small deviator from the viscous fluid. Thus the scheme considered here assumes that the viscoelastic flow is perturbed about the viscous fluid. The contribution of the elasticity is included in the first-order problem. Due to the linear stability theory we stop at the first-order perturbation expansion. In the nonlinear stability we can go to higher orders of the problem. The nonlinearity of the problem at hand will not be discussed here and it will be the subject of a subsequent paper.

The numerical computations show that the kinematic viscosity plays a stabilizing role in the zero-order problem, i.e in the absence of the elasticity ($\lambda = 0$) and opposite role in the first-order problem, i.e in the presence of λ. Thus the viscosity has changed its mechanism in the non-Newtonian state to play the same destabilizing role that the Maxwell relaxation time λ is playing.

Appendix A

The governing equations of the zero- and first-order problems are listed below.
(I) The set of the zero-order equations are:

$$\left(\frac{\partial^2}{\partial y^2} - k^2\right)\left[\nu^{(r)}\left(\frac{\partial^2}{\partial y^2} - k^2\right) - D_0\right]\hat{V}_0^{(r)} = 0 \tag{A1}$$

$$\frac{\partial}{\partial y}\hat{\pi}_0^{(r)} = \rho^{(r)}\left[\nu^{(r)}\left(\frac{\partial^2}{\partial y^2} - k^2\right) - D_0\right]\hat{V}_0^{(r)} \tag{A2}$$

$$\hat{V}_0^{(1)} = \hat{V}_0^{(2)} = D_0\gamma_0 \qquad y = 0 \tag{A3}$$

$$\frac{\partial}{\partial y}[\hat{V}_0^{(1)} - \hat{V}_0^{(2)}] = 0 \qquad y = 0 \tag{A4}$$

$$\left(\frac{\partial^2}{\partial y^2} + k^2\right)[\nu^{(1)}\rho^{(1)}\hat{V}_0^{(1)} - \nu^{(2)}\rho^{(2)}\hat{V}_0^{(2)}] = 0 \qquad y = 0 \tag{A5}$$

$$\hat{\pi}_0^{(1)} - \hat{\pi}_0^{(2)} + \tilde{\sigma}\gamma_0 - \mathrm{i}kE_0(\varepsilon^{(1)}\phi_1^{(1)} + \varepsilon^{(2)}\phi_1^{(2)}) - 2\nu^{(1)}\rho^{(1)}\frac{\partial}{\partial y}\hat{V}_0^{(1)} + 2\nu^{(2)}\rho^{(2)}\frac{\partial}{\partial y}\hat{V}_0^{(2)} = 0$$
$$y = 0. \tag{A6}$$

(II) The set of the first-order equations are:

$$\left(\frac{\partial^2}{\partial y^2} - k^2\right)\left[\nu^{(r)}\left(\frac{\partial^2}{\partial y^2} - k^2\right) - D_0\right]V_1^{(r)} = \left(\frac{\partial^2}{\partial y^2} - k^2\right)(D_1 + \tilde{\lambda}^{(r)}D_0^2)\hat{V}_0^{(r)} \tag{A7}$$

$$\frac{\partial}{\partial y}\hat{\pi}_1^{(r)} = \rho^{(r)}\left[\nu^{(r)}\left(\frac{\partial^2}{\partial y^2} - k^2\right) - D_0\right]\hat{V}_1^{(r)} - \rho^{(r)}(D_1 + \tilde{\lambda}^{(r)}D_0)\hat{V}_0^{(r)} - \tilde{\lambda}^{(r)}D_0\frac{\partial}{\partial y}\hat{\pi}_0^{(r)} \tag{A8}$$

$$\hat{V}_1^{(1)} = \hat{V}_1^{(2)} = D_1\gamma_0 + D_0\gamma_1 \qquad y = 0 \tag{A9}$$

$$\frac{\partial}{\partial y}[\hat{V}_1^{(1)} - \hat{V}_1^{(2)}] = 0 \qquad y = 0 \tag{A10}$$

$$\left(\frac{\partial^2}{\partial y^2} + k^2\right)[\rho^{(1)}\nu^{(1)}\hat{V}_1^{(1)} - \rho^{(2)}\nu^{(2)}\hat{V}_1^{(2)}] = 0 \qquad y = 0 \tag{A11}$$

$$\hat{\pi}_1^{(1)} - \hat{\pi}_1^{(2)} + kE_0\varepsilon^*\gamma_1 + \tilde{\sigma}\gamma_1 - 2\rho^{(1)}\nu^{(1)}\frac{\partial}{\partial y}\hat{V}_1^{(1)} + 2\rho^{(2)}\nu^{(2)}\frac{\partial}{\partial y}\hat{V}_1^{(2)} + 2\sigma_0\rho^{(1)}\nu^{(1)}\tilde{\lambda}^{(1)}\frac{\partial}{\partial y}\hat{V}_0^{(1)}$$
$$- 2\sigma_0\rho^{(2)}\nu^{(2)}\tilde{\lambda}^{(2)}\frac{\partial}{\partial y}\hat{V}_0^{(2)} = 0 \qquad y = 0. \tag{A12}$$

Appendix B

To separate the real and imaginary parts of equation (76). If we take

$$\sigma_0 = x + \mathrm{i}y \tag{B1}$$

then m^*, which is given by relation (74), takes the following form:

$$m^* = a + \mathrm{i}b \tag{B2}$$

where

$$a = \frac{1}{\sqrt{2}}\left\{\left(1+\frac{x}{k^{*^2}\nu^*}\right)+\left[\left(1+\frac{x}{k^{*^2}\nu^*}\right)^2+\left(\frac{y}{k^{*^2}\nu^*}\right)^2\right]^{1/2}\right\}^{1/2} \tag{B3}$$

$$b = \frac{y}{\sqrt{2}}\left\{\left(1+\frac{x}{k^{*^2}\nu^*}\right)+\left[\left(1+\frac{x}{k^{*^2}\nu^*}\right)^2+\left(\frac{y}{k^{*^2}\nu^*}\right)^2\right]^{1/2}\right\}^{-1/2}. \tag{B4}$$

Substituting from (B1) and (B2) into (69) and (70) and considering the case of $\nu^{(1)} = \nu^{(2)} = \nu$ for separating the real and imaginary parts, we obtain:

$$p_1^* = x(x^2-3y^2)(1+\rho)^2+k^{*^2}\nu^*\{(x^2-y^2)[5(1-\rho)^2+8\rho]-[a(x^2-y^2)-2bxy][2(1-\rho)^2+5\rho]\}$$
$$+k^{*^4}\nu^{*^2}\{x[5(1-\rho)^2+2\rho]-(ax-by)[7(1-\rho)^2+4\rho]\}+6k^{*^6}\nu^{*^3}(1-a)(1-\rho)^2 \tag{B5}$$

$$q_1^* = y(3x^2-y^2)(1+\rho)^2+k^{*^2}\nu^*\{2xy[5(1-\rho)^2+8\rho]-[b(x^2-y^2)+2axy][2(1-\rho)^2+5\rho]\}$$
$$+k^{*^4}\nu^{*^2}\{y[5(1-\rho)^2+2\rho]-(ay+bx)[7(1-\rho)^2+4\rho]\}-6k^{*^6}\nu^{*^3}b(1-\rho)^2 \tag{B6}$$

$$\begin{aligned} p_2^* = \lambda\rho\{&2(x^4-6x^2y^2+y^4)(2-a-\rho-\rho^2)+8bxy(x^2-y^2)\\ &+k^{*^2}\nu^*(7\rho^2+4\rho-11)[ax(x^2-3y^2)-by(3x^2-y^2)]\\ &-2x(x^2-3y^2)(8\rho^2+6\rho-11)+2k^{*^4}\nu^{*^2}[(11\rho^2+2\rho-13)[a(x^2-y^2)-2bxy]\\ &-(x^2-y^2)(15\rho^2+2\rho-17)]+k^{*^6}\nu^{*^3}(1-\rho^2)(16-ax+by)\}\\ &+2(x^4-6x^2y^2+y^4)[(2-a)\rho^2-\rho-1]+8\rho bxy(x^2-y^2)+k^{*^2}\nu^*\{x(x^2-3y^2)\\ &\times[11(2-a)\rho^2+4(a-3)\rho+7a-16]+by(3x^2-y^2)(11\rho^2-4\rho-7)\}\\ &+12k^{*^4}\nu^{*^2}\{(17\rho^2-2\rho-15)(x^2-y^2)-(13\rho^2-2\rho-11)[a(x^2-y^2)-2bxy]\}\\ &+k^{*^6}\nu^{*^3}(1-\rho^2)(ax-by-16) \end{aligned} \tag{B7}$$

$$\begin{aligned} q_2^* = \lambda\rho\{&-2[4axy(x^2-y^2)+b(x^4-6x^2y^2+y^4)]-8xy(x^2-y^2)(\rho^2+\rho-2)\\ &+k^{*^2}\nu^*2y(3x^2-y^2)(11-6\rho-8\rho^2)+(7\rho^2+4\rho-11)[ay(3x^2-y^2)\\ &+bx(x^2-3y^2)]]+2k^{*^4}\nu^{*^2}[(11\rho^2+2\rho-13)[2axy+b(x^2-y^2)]\\ &-2xy(15\rho^2+2\rho-17)]-k^{*^6}\nu^{*^3}(1-\rho^2)(ay+bx)\}-2\rho^2[4axy(x^2-y^2)\\ &+b(x^4-6x^2y^2+y^4)]+8xy(2\rho^2-\rho-1)(x^2-y^2)+k^{*^2}\nu^*\{2(3x^2-y^2)\\ &\times(11\rho^2-6\rho-8)-(11\rho^2-4\rho-7)[ay(3x^2-y^2)+bx(x^2-3y^2)]\}\\ &+2k^{*^4}\nu^{*^2}\{2xy(17\rho^2-2\rho-15)-(13\rho^2-2\rho-11)[2axy+b(x^2-y^2)]\}\\ &+k^{*^6}\nu^{*^3}(1-\rho^2)(ay+bx) \end{aligned} \tag{B8}$$

$$\lambda = \lambda^{(1)}/\lambda^{(2)}. \tag{B9}$$

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